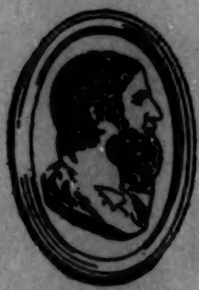
THE MATHEMATICS TEACHER



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THE MATHEMATICS TEACHER

Devoted to the interests of mathematics in Elementary and Secondary Schools

This organization has for its object the advancement of mathematics teaching in junior and senior high schools. All persons interested in mathematics and mathematics teaching are eligible to membership. All members receive the official journal of the National Council—THE MATHEMATICS TEACHER—which appears monthly, except June, July, August and September.

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WILLEBRORDUS SNELLIUS.
PROFESSOR MATHESEOS.

THE MATHEMATICS TEACHER

Volume XXIV — Number 4

Edited by William David Reeve

"The Mathematics Club Meets"*

By WILIMINA EVERETT PITCHER
Rawlings Junior High School, Cleveland, Ohio

CAST OF CHARACTERS

ERNEST, club president. Later, poses as Napier.
PAULINE, club secretary, in charge of club program.
WINIFRED, acting secretary.
HENRIETTA, ROSE — club members. They also pose for little French girls.
CARMELLA, club member. Also poses as Egyptian.
GAZZIE, club member. Poses as Roman.
GLEN, announcer. A bored club member.
MARY, IRENE, CAROLYN, LUCILLE, MARTIN, IRENE RUTH, GUS, LENA — other club members.

* Presented by the Mathematics Club of Rawlings Junior High School, Cleveland, January, 1931.

Properties

Large picture frame.

Small easel or support of some sort for "stone tablet." Stone tablet may be drawn on cardboard. A good picture is given in Smith: *History of Mathematics,* Volume II, page 46.

A large chair for Napier.

Setting

A well-equipped mathematics classroom. In center of room a large picture frame, behind which the pictures can be posed.

An easily drawn curtain should hang in front of the frame.

Costumes

A long white or colored robe for the Roman.

A short, scantily cut, one-piece white garment and matching headdress for the Egyptian.

An academic gown, white ruff, shaggy beard for Napier.

French peasant costumes for little French girls.

Announcement

The Rawlings Math Club adopted as its project this fall the study of early methods of writing numbers and ways of adding, subtracting, and multiplying with such numbers. We did not choose this work because we were interested in it, but because our sponsor said it would make an interesting assembly, and we were determined to give an assembly. We have, therefore, learned to multiply as people did when multiplication was an almost impossible task for most people.

We have had so much fun doing these things, that today we are going to give you the opportunity of enjoying some of them with us. However, we can show you only a few of the ways in which people added. We mention only one of the ways by which the Greeks wrote numbers. Our time is too limited for us to attempt more. The assembly which the Math Club presents to you today is a one-act play called *The Mathematics Club Meets.*

(PAULINE, *notebook in hand, enters auditorium from west door and starts to walk across the front of the auditorium.* ROSE, WINIFRED, *and* HENRIETTA, *carrying books, hurry in after her.)*

WINIFRED. *(calling)* O Pauline, wait for us.

PAULINE. *(turning)* Hello, girls. Hurry! You know there is a lot to do.

ROSE. We know there is, and aren't you excited? I could hardly wait for club period to come. I thought those first two periods would never end. I'm so glad Mrs. Pitcher was able to get those pictures to show us.

WINIFRED. Minutes ready, Pauline?

PAULINE. Yes they are, and Winifred, be a good sport and read them for me. I have plenty to do without them. Here they are. I hope you can read them. *(Points them out carefully in the note-book which she passes to Winifred. Girls go up on stage just as another group enters from the other side. The two groups exchange greetings.)*

HENRIETTA. *(looking about)* Where's Ernest?

ROSE. Wouldn't you just know he'd be late on a day like this?

Ernest enters

ROSE. It's about time.

IRENE RUTH. Where were you? What teacher kept you?

ERNEST. I guarded last period. I had to take a guy to the office. Mrs. Pitcher was there, too. Mr. Porter sent for her. Something about a boy in her home room. She said we'd have to get along without her today. *(Looking about)* The meeting will now come to order. *(Waits while children settle in seats)* The secretary will read the minutes of the last meeting.

WINIFRED. Tuesday, January 13, 1931. The meeting was called to order by the president at 9:45. The minutes were read and adopted as read. The meeting was then turned over to the sponsor, Mrs. Pitcher.

We had a review of finger notation by our great mathematician, Gus. We took our rods and multiplied some very large numbers. Our motto is "Multiplication is vexation," but we didn't have a hard time doing the multiplication.

Mrs. Young visited our club and it was explained to her how we could use these rods. Mrs. Young is a math teacher, but she doesn't know how to multiply anything larger than five times five so we showed her how. She likes the work we are doing and she said she is going to visit us again. The club was adjourned at 10:30.

Pauline Mason, secretary

ERNEST. Any business to come before the club today? *(Pause)* If not, I'll turn the meeting over to Pauline Mason who has charge of today's program.

PAULINE. *(rising)* Mrs. Pitcher has borrowed some pictures to show us today. They all tell something about the curious mathematics we have studied this year in the club. We are going to look at the pictures and see how much we can remember about the stories that go with them. You better push back your chairs and stand here and here so all can see better. *(Pulling back curtain)* Now this is a picture of an early Egyptian and the kind of figures he used when he wanted to write numbers.

HENRIETTA. Oh, yes! I'll always remember them because I think hieroglyphics is such a funny name to call figures by.

IRENE. Wonder what those are written on?

IRENE RUTH. I know. I read a story about Egyptians just last week. They wrote on papyrus—

IRENE. What's papyrus?

IRENE RUTH. It was a kind of paper. They made it from a reed that grew along the banks of the Nile. They used a reed pen, too. They could get plenty of reeds, and they used papyrus for all long records. Sometimes they wrote on wood or broken pottery and they carved numbers on stone if they wanted them to last a long time. I bet that's a stone tablet. *(Children nod vigorously)*

PAULINE. *(pointing)* This number is one. They made numbers up to ten just by using enough ones. This is ten. They grouped ones and tens for figures to one hundred. There is one hundred.

HENRIETTA. *(pointing)* Wonder what that old Egyptian could do with his numbers?

PAULINE. Oh, add, and maybe subtract. I don't think he could multiply or divide, though. You remember what a hard time people were having with multiplication and division six or seven hundred years ago. That Egyptian may have lived 4,000 years ago. Just remember that multiplication and division were hard.

MARTIN. I'll say! I missed all my division yesterday just on account of an old decimal point. I wish I could have seen that old Egyptian add, though.

GUS. Huh. You could now if you had any imagination. Just look at him hard. I see him add.

LENA. What's he going to add, Gus?

GUS. Well, let's see. Give him something easy and a nice clean papyrus. *(Changes paper on stone as he talks)* Now, Ahmes—

HENRIETTA. Oh, Gus, is he really Ahmes?

GUS. I don't know, but I can call him Ahmes if I want to. Now, Ahmes, add thirty-two and twenty-six.

LENA. Why I see him too, Gus. Doesn't he write funny? He puts the tens on the right.

ERNEST. Number order meant nothing to an Egyptian. He wrote from left to right, or from right to left. Sometimes he even wrote the tens above the units. I think, though, that the best educated Egyptians started at the right.

MARY. This Egyptian's good. Do you think we could make him add if he had to carry?

PAULINE. Of course he could. Just imagine him adding seventy-five and forty-six. *(Egyptian writes numbers and points slowly)*

MARY. *(counts)* One, two, three, four, five, six, seven, eight, nine, ten. *(Egyptian reaches ten and stops pointing to write ten)*

LUCILLE. Oh, he carried ten!

PAULINE. Didn't I say he could? Now watch him finish. He wrote that one just as we would. I like the way he carries numbers. One hundred twenty-one is the correct answer, too. Let's see another picture. *(Draws curtain)*

LENA. I'm glad we don't write numbers that way. It takes too long. The addition is easier than ours, though. No number combinations to remember. Just count figures.

GUS. I'd like to see an old Greek add.

ERNEST. Yes, just because you're Greek. Well, I don't think their numbers were interesting. They just used the letters as they came in the alphabet and put a line over them so they would know they were numbers and not letters. Nothing to that. I don't think the Greeks knew much about math.

GUS. Oh, they didn't? What about Euclid?

ROSE. Euclid—geometry—*ugh!*

Carmella enters

LUCILLE and LENA. What made you so late?

CARMELLA. The dentist sent for me and he wants Henrietta and Rose right away. *(Girls leave)* Say, Gus, I heard you trying to brag about the Greeks. Well, just remember that story about Italian mathematics.

LUCILLE. What was it?

CARMELLA. A wealthy German who lived in the fifteenth century had a son who wanted to be a merchant. The father wanted his son to be a very good merchant, so he asked a German college professor to tell him the name of a good college for his son. The professor said, "If you want him to learn to add and subtract send him to any German university. They are all good. If you want him to learn difficult mathematics, such as multiplication and division, send him to an Italian university. The Italians are the only people who teach such difficult mathematics."

GUS. Oh, Carmella, think about Euclid, and Plato, and Pythagoras, and—

PAULINE. You better shut up and look at the rest of these pictures. You know we can't have them next week. *(Draws curtain)* Now here is an old Roman and his addition with Roman notations.

CAROLYN. Oh, let me try it. L is fifty; those three X's are tens; V is five; and I is one. The top number is eighty-six. *(Children nod approval)*

IRENE. And the bottom number is thirty-two. He didn't carry tens, though. He carried fives.

CAROLYN. *(Roman slowly does addition while Carolyn talks)* Two and one is three. Write the three. There is nothing to add to five so he writes the five. Ten, twenty, thirty, forty, fifty, and one more ten. Of course he wrote ten and carried fifty. Fifty and fifty make one hundred. He has to carry that so he has nothing to write in the fifty column. He puts C in the hundred column. I suppose that is easier than our addition especially if your math is so poor you can't count well above five.

PAULINE. *That's* not such a good joke. You know very well, or you ought to, that for a long time lots of people couldn't count even so far as five. Wait till you see the next picture. Oh, and did you see this old book? *(Passes it around)* It has such funny looking letters in it.

IRENE. I wish it were an arithmetic, because then it might have a really old picture of finger notation in it. Lots of arithmetics had finger notation even as late as the sixteenth century.

GUS. I can make good numbers on my fingers. I read about them in a book Mrs. Pitcher had on her desk. It said finger notation had to be given in arithmetics because all nations wrote figures in a differ-

ent way. All traders, though, knew how to make numbers on their fingers. They had to, or they couldn't trade with each other. With the left hand they made numbers less than one hundred. They used these three fingers for numbers below ten and these two for tens. Let Winifred and me show you. Winifred makes good numbers, too. *(Two children make numbers, naming them)* The numbers above one hundred were just like those below, except that they were made on the other hand. They went like this. *(Makes numbers)* *(Rubs his hand)* That's fine finger exercise if you play a piano or run a typewriter. *(Gazzie enters. Glen, who has been sitting off by himself, doing homework, looks up)*

GLEN. Where were you? Why didn't you get here on time? *(After Gazzie answers, Glen goes back to his homework)*

GAZZIE. I can't help it. I went on an errand for Miss Perner. Miss Miller says Mr. Porter wants Ernest Pastor right now.

ERNEST. *(starts out grumbling)* Did she say right now? Why can't he wait till next period? I have wood shop next period.

PAULINE. *(drawing curtain)* Here is another picture. Who knows what they are doing?

GAZZIE. Oh, I do. Let me tell. They are little French girls, and they are doing a multiplication problem. You know we learned that many French peasants still multiply on their fingers. Lots of people in Europe used finger multiplication hundreds of years ago. Multiplication was hard.

MARTIN. I say it is!

GAZZIE. For finger multiplication they only needed to know the multiplication table to five times five and be able to add a little. Of course they couldn't multiply big numbers.

LUCILLE. You mean they didn't have to know six times nine and seven times eight? I always mix those two. Let's all think hard and see if we can make them do problems for us as the Egyptian and Roman did. *(Children take what they think are "concentrating" poses. French girls slowly close their hands)*

IRENE. Let's pretend they are going to multiply seven times eight for Lucille. *(French children put up two fingers)*

MARTIN and LUCILLE. *(excitedly)* Look! Look!

IRENE. Five from seven is two so they put up two fingers. Five from eight is three, so they put up three fingers. The fingers standing are tens. *(Pointing)* She has ten, twenty, thirty, forty, fifty, on the

fingers that stand up. She has two fingers down on one hand for units and three on the other. They have to multiply those, but they can, because they know their tables to five times five. Two times three equals six. Fifty plus six is fifty-six. Remember that, Lucille, and seven times eight won't bother you.

LUCILLE. Wonder if I can do six times nine. *(As she does the multiplication, the French children in the picture do the same thing)* Five from six is one. One finger up. Five from nine is four. Four fingers up. Ten, twenty, thirty, forty, fifty. One times four is four. Fifty plus four equals fifty-four. Gee! I hope I remember that.

IRENE RUTH. Wonder if the little French girls can multiply fourteen times twelve. The rule is different and harder.

PAULINE. Of course they can. They look very intelligent. Just watch them.

LENA. Oh, let me help them. *(French children and Lena do multiplication)* Ten from twelve leaves two. Two fingers up. Ten from fourteen leaves four. Four fingers up. Count the fingers standing for tens. Ten, twenty, thirty, forty, fifty, sixty. Multiply the fingers standing for units. Two times four is eight. Sixty plus eight is sixty-eight. Add it to one hundred and you get one hundred sixty-eight for an answer, and if you don't believe that's right you'll have to multiply it out on paper.

PAULINE. Are you ready for another picture?

MARY. Please, Pauline, not yet. I want to tell Lucille a poem that my mother taught me. She said she learned it to say at school on Friday afternoon, when she was a little girl. They always spoke pieces on Friday afternoon.

LUCILLE. Let her tell me, Pauline. *(Pauline nods)*

MARY. Six Times Nine.

I studied my tables over and over
And backward and forward too.
But I couldn't remember six times nine
And I didn't know what to do.

But sister told me to play with my doll
And not to bother my head.
"If you call her Fifty-four for a while,
You'll learn it by heart," she said.

So I took my favorite, Mary Ann,
Though I thought 'twas a dreadful shame
To call such a perfectly lovely child
Such a perfectly horrid name.

But I called her my dear little Fifty-four
A hundred times till I knew
The answer to six times nine as well
As the answer to two times two.

Next day Elizabeth Wigglesworth
Who always acts so proud
Said, "Six times nine is fifty-two"
And I nearly laughed aloud.

But I wished I hadn't when teacher said,
"Now, Dorothy, tell if you can,"
For I thought of my doll, and sakes alive,
I answered, "Mary Ann."

MARTIN. That's a good poem for girls, but what I want is a way to multiply where you don't need any multiplication tables at all. I never remember them.

LUCILLE. You ought to use the Russian peasant multiplication, Martin.

MARTIN. What's that?

LUCILLE. I guess you were absent the day we had it, but I can show you how it goes. You have to know the table of two's though, and be able to divide by two and add well.

MARTIN. I knew there was a catch in it, somewhere.

LUCILLE. They say the peasants of Russia still use this method and get to be very quick at it. I'll multiply twenty-five by forty-eight. *(Writes numbers)* I'm going to divide the numbers in this column and multiply these. It doesn't make any difference which way I start though. *(Lucille writes and talks at same time)* Twenty-five divided by two is twelve and you don't have to bother about remainders.

MARTIN. Suits me.

LUCILLE. Forty-eight times two is ninety-six. Twelve divided by

two is six. Ninety-six multiplied by two is one hundred ninety-two. Six divided by two is three. One hundred ninety-two times two is three hundred eighty-four. Three divided by two is one. Two times three hundred eighty-four is seven hundred sixty-eight, and since we can't divide any more we are through dividing and multiplying. Even Martin will have to admit that that wasn't hard. Cross out the numbers opposite the even quotients. I suppose, Martin, you know what a quotient is. Add the products that are left. *(Adds rapidly, mentally)* The answer is twelve hundred.*

Martin. Well, that's a good method, but what would Mr. Miller say if he caught me using it?

Carmella. Huh, he'd be glad to see you get a multiplication problem right, just once, no matter what method you used.

Martin. I wish we had invited all the math teachers to this meeting. Maybe, if they could see what a hard time people had with the beginning of arithmetic, they wouldn't holler about our work.

Pauline. I wish we had invited them. *(Drawing curtain)* Now this picture is that old Scotchman, Napier, that Mrs. Pitcher said worked twenty years on a problem. How's that for homework? I asked her what the problem was, and she said he was trying to make multiplication and division easier for people. He succeeded, too. He called his problem logarithms. I asked Mrs. Pitcher what logarithms were and she said exponents.

Martin. What are exponents, Pauline?

Irene. Aw shut up. You wouldn't know even if Pauline told you. It's something the 9A's are always talking about when they do their algebra. Glen says it's hard. *(Glen looks up and nods)*

Pauline. Napier also invented a very simple multiplication machine, the napier's rods that we made in club. Let's get the big set and do some problems for Mr. Napier. Someone has to tell the story though. *(Hands go up)*

Pauline. I'll let Irene Ruth do it.

* 25	48
~~12~~	~~96~~
~~6~~	~~192~~
3	384
1	768
	1200

IRENE RUTH. People who had to do much multiplication made sets of rods which they carried about with them. Sometimes they made them of wood, but very often they made them of strips of paper about three and one-half inches long. This rod has on it the table of nines. People didn't have to know the table. They could copy it. Here is one times nine, two times nine, three times nine, etc. This is the table of sixes. This rod has just numbers on it. It is the multiplier rod. Let's multiply 4,296 by 123. The first thing they needed to do was lay out the multiplicand. Here, some of you kids, help. *(Places rods)* Pauline will write for us. Here is the multiplier rod. Will you hold it, Mary? Now, Pauline, write 123. They didn't write the multiplicand because they had it laid out where they could see it. They had to write the multiplier so they wouldn't forget it. The product by three is eight, eight, eight, two, one; by two is two, nine, five, eight, and by one is six, nine, two, four. Adding eight, zero, four, eight, two five. Martin wouldn't even need to learn the two's to do that. Let's try another one.

PAULINE. I wish we could but it's time for the bell, so we'll have to stop, now. If you want to see the rest of the pictures come to 216 tonight after school. *(Children start out)* *(Curtain)*

NOTE: The following four books will be found helpful to anyone who cares to follow out any of the topics taken up in this play.

1. DANTZIG, TOBIAS: *Number the Language of Science,* Macmillan, New York, 1930. Story of the German Merchant, p. 26; French peasant multiplication, p. 11.
2. SANFORD, VERA: *A Short History of Mathematics.* Houghton Mifflin Company, Boston, 1930, pp. 76, 77, 81, 202, 203.
3. SMITH, DAVID EUGENE: *History of Mathematics,* Ginn and Company, Boston, 1925, pp. 45-53, 54-64, 196-202, 202-203.
4. SMITH, DAVID EUGENE: *Source Book in Mathematics,* McGraw-Hill Book Company, New York, 1929, 182-186.

Significant Digits in Computation with Approximate Numbers

By E. A. Bond
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Results of operations involving measured quantities are frequently expressed with an indicated degree of precision far beyond that which the approximate nature of the measurements warrants. This is especially true in textbooks both in numerical mathematics and in statistics. Accordingly, it seems to the writer that a study should be made of the number of significant figures that should be retained to express the results of each of the operations involving approximate numbers. This article is an effort to contribute to this problem.

One can find in the textbooks in current use in the fields mentioned convincing evidence of the truth of the statement of the last paragraph by a comparison of the number of figures retained in the results with the degree of precision in the measurements. The following illustrations of cases of this practice are from textbooks in use at the present time:

1. In an eighth grade mathematics book, published within the last two years, the contents in gallons of a well is computed to six significant figures. Four of these are submitted as the answer to the problem, although the measurements of the dimensions were of the crudest kind. Moreover the diameter was expressed with one significant figure and the depth with two. Furthermore the assumption was implied that the well was a true right cylinder uniform in diameter and level upon the bottom. In addition to this the approximation of $7\frac{1}{2}$ gallons to the cubic foot was used. Accordingly a constant error of computation of one-half of one per cent too much was introduced. The contents should have been expressed with one significant figure instead of four.

2. In another junior high school textbook in mathematics the result of a continued multiplication is expressed with nine significant figures while the approximate nature of two of the factors was such that the only figure that is reliable is the extreme left-hand one.

3. In textbooks in statistics one frequently finds the mean, the median and the standard deviation expressed with three or four decimal places when the measurements or groupings are such that these measures are reliable only to the nearest unit. For example, in a widely used textbook in statistics the maximum temperature is taken to the nearest degree for 62 days. The mean of these readings is computed to three decimal places, the median to four and the standard deviation (sigma) is expressed with three decimal places. Now the assumption back of the procedure is that there was the same amount of overreadings as underreadings. This assumption would be valid only if a very large number of readings were taken, and if no constant error were present. There would be two sources of constant error that would enter into the readings: first, the lag of the mercury would cause a constant underreading and, second, the constant tendency of the observer to read too high or too low. Furthermore a mean temperature expressed to the thousandth of a degree is purely a fancy and as such can serve no useful purpose.

Let us consider for a moment the meaning usually attached to significant figures. These are the figures that remain after the ciphers that are used to locate the decimal point are disregarded. The following examples should make that definition clear:

1. When the velocity of light is expressed as 186,000 miles per second it is expressed with three significant figures and indicates that this is the velocity to the nearest 1,000 miles per second.

2. The ratio of the inch to the centimeter is expressed with three significant figures as 2.54 and with four as 2.540.

3. The ratio of the circumference of a circle to its diameter (π) is 3.1416 expressed with five significant figures and 3.14 with three.

4. The wave length of the D line in the sodium flame is 0.00005896 cm. expressed with four significant figures and as 0.000059 cm. expressed with two.

Final zero in a measured whole number may be significant. For example, one could measure a length of 20 centimeters and express it as 20 cm. On the other hand 1,840.0 ft. indicates that the distance has been measured to the nearest 0.1 ft. In the last case the distance is expressed with five significant figures.

Since it is always possible in advance of performing the operation of multiplication or of division to determine the degree of precision that is either desired or consistent with the measurements, the work

may be greatly abbreviated by the use of contracted multiplication or division. For example, if the diameter of a circle has been measured to the nearest 0.01 of a centimeter and expressed with four significant figures as 27.34 cm. the circumference should not be expressed with more than four significant figures. It would, therefore, in this case be carried to the nearest 0.01 cm. The result is found by contracted multiplication in the margin.

```
27.34
 3.1416
-------
82 02
 2 73
 1 09
    3
    2
-------
85.89
```

Even in this case the right-hand figure of the product would not be reliable since the error that was made in the measurement of the diameter is multiplied by a little more than 3 in the product. Hence the circumference should be expressed as 85.9 cm.

Let us test the result of the procedure in the last paragraph and see between what limits the product really lies. A diameter of 27.34 cm. lies between 27.335 cm. and 27.345 cm. Using each of these and a value of π consistent with them the product is found for both cases in the margin.

```
27.335     27.345
 3.1416     3.1416
-------    -------
82 005     82 035
 2 734      2 735
 1 093      1 094
    27         27
    16         16
-------    -------
85.875     85.907
```

The results indicate that the circumference should be expressed as 85.9 cm. since they agree thus far. Usually it is safer to carry the result of multiplication or of addition one place farther than is desired. Then the last digit should be dropped and one added to the last digit retained if the figure dropped is 5 or more.

When both factors in a multiplication are measured numbers, there is a double error possible both of which should be taken into account. The area of a rectangle whose dimensions are 76.34 ft. and 54.28 ft. if carried to the nearest .01 sq. ft. is found in the margin.

```
  76.34
  54.28
-------
 305 36
3817 00
  15 27
   6 11
-------
4143.74
```

It can however be shown that the two right-hand figures are unreliable. Hence the result that should be submitted in this case is 4,144 sq. ft. The number of square feet of area lies between 76.335×54.275 and 76.345×54.285. That is, they lie between 4,143.08 and 4,144.39. Hence when the area is expressed as 4,144 sq. ft. the right-hand figure is in doubt. However, 4,144 sq. ft. is nearer the truth than 4,140 sq. ft. At any rate a result expressed with more than four significant figures would be inconsistent with the data.

76.335	76.345
54.275	54.285
305 34	305 38
3816 75	3817 25
15 27	15 27
5 34	6 11
38	38
4143.08	4144.39

Generally, it can be said that the result of a multiplication involving approximate numbers should not be expressed with more significant figures than the factor that has the fewer significant figures contains.

The sum of a number of addends is unreliable to the right of the right-hand figure of the least exact one. Hence before adding the addends may be "rounded off" to the number of places that the least exact one contains. Accordingly examples such as that in the margin are pure fancy and should not be found in tests or exercise books. Unless these addends are each measured to the nearest 0.001, the sum should not be expressed to that place. If on the contrary they are all measured to the nearest 0.001, each addend should be expressed to thousandths as in the margin. Even here the right-hand digit of the sum is doubtful since each addend is an approximation which may differ either way from the true magnitude by as much as one-half of a thousandth. Hence the sum should be expressed as 21.74 with the reservation that the right-hand 4 may be a 3.

6.826
7.52
6.8
.503
.087
21.736

6.826
7.520
6.800
.503
.087
21.736

If, on the other hand, the 6.8 is measured only to the nearest 0.1 of a unit the addition and the sum is as indicated. Of course there are cases when the addends are exact. For example when $3\frac{1}{2}$, $2\frac{3}{4}$ and $4\frac{5}{8}$ are reduced to decimals and added, the supplying of zeros is a matter of convenience and not a necessity. In the foregoing discussion I have reference to measured quantities.

6.8
7.5
6.8
.5
.1
21.7

3.5
2.75
4.625
10.875

In the subtraction of two measured numbers, each measured to the same degree of precision, the difference between the maximum and the minimum differences is two units of the right-hand place.

63.65	63.75
61.45	61.35
2.20	2.40

Thus, the difference between the heights of two boys expressed as 63.7 in. and 61.4 in. lies between 2.2 in. and 2.4 in. Hence the difference is 2.3 in. $\pm$ 0.1 in.

In the division of an approximate number the quotient should not be carried out to more significant fig-

```
          9.4
3.1416)29.6
       28 3
        1 3
```

ures than the dividend contains. For example, if the circumference of a tree has been measured to the nearest 0.1 ft. as 29.6 ft. the diameter should not be expressed with more than three figures. In this case the third is doubtful. Hence the diameter should be expressed as 9.4 ft. Since it lies between 9.41 ft. and 9.44 ft.

```
          9.41                 9.44
3.1416)29.55         3.1416)29.65
       28 27                28 27
        1 28                 1 38
        1 26                 1 26
           2                   12
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          3.916                3.921
9.865)38.635         9.855)38.645
      29 595               29 565
       9 040                9 080
       8 879                8 870
         161                  210
          99                  197
          62                   13
```

If the dividend and the divisor are both the results of measurements there is a double source of error. For example, if 38.64 is to be divided by 9.86 and if they are both the results of measurements the quotient is 3.92 true to these three significant figures. Moreover additional figures would not be reliable, since the true quotient lies between 39.635 $\div$ 9.865 and 38.645 $\div$ 9.855. Hence, it lies between 3.916 and 3.921. It is, therefore, 3.92, since there is agreement thus far.

From the foregoing discussion it is evident that in operations involving approximate numbers the following rules are valid:

I. The sum of a number of addends should not be expressed with more significant figures than the least accurate addend contains.

II. The difference between two numbers should not be expressed with more significant figures than the less accurate one contains.

III. The product of two numbers should not contain more significant figures than the less accurate factor contains.

IV. The quotient of two numbers should not contain more significant figures than the less accurate one of these contains.

Notes on the First Year of Demonstrative Geometry in Secondary Schools

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For most pupils in our secondary schools the first year of demonstrative geometry is the only year of demonstrative geometry. A great many schools devote the whole of Grade X to this subject and some return to it in a later year. The course of study in mathematics could probably be arranged so that no school need give more than one academic year, or the equivalent, to this first course in demonstrative geometry. One way of doing this would be to teach the geometry parallel with the algebra in the tenth and eleventh grades, giving the geometry Monday, Tuesday, and Wednesday of each week in Grade X, and twice a week in Grade XI. There would probably be no algebra in the opening weeks of Grade X until the geometry is well started.

Probably not more than one-tenth of all the pupils in the United States who take a first year of demonstrative geometry go on to the study of solid geometry.* Even in the strongest schools this fraction rarely exceeds one-sixth. It seems clear that some study of the geometry of three dimensions has value for almost all pupils of secondary school age, but that few of them will have this advantage unless we make definite provision for it either in the informal geometry of the junior high school years, or in the first year of demonstrative geometry. I see good reason to include it in both, putting as much as possible of the three-dimensional work into the informal geometry of the junior high school, and fitting the remainder into the first year of demonstrative geometry. Much of the three-dimensional material that I should wish to see rescued for future generations is but an extension of certain portions of the two-dimensional material commonly offered in the tenth grade. Even though I do not recommend rigorous logical treatment of this three-dimensional material, I think it ought to be presented in connection with its two-dimensional analogues. If

* Inferred from the *Bureau of Education Bulletin,* 1924, No. 7, pp. 46, 47.

some future reorganization of the course of study should relegate some of this two-dimensional material to the junior high school years for informal treatment, then the corresponding material in three dimensions would naturally accompany it in this revised program.

Although the present reconsideration of the course of study in geometry arose from our concern for the three-dimensional aspect of the subject, and how to save it, there may well be other points that deserve attention also. For example, we are probably mistaken in our practice of asking beginners to prove the obvious. They would probably catch the spirit of demonstrative geometry much quicker if we were to assume at the outset a much longer list of propositions than we do at present. I believe that most students would be glad to take for granted the following:

*Assumptions for Book I**

1. *There is one and only one straight line through two given points.*
2. *Two distinct straight lines cannot have more than one point in common.*
3. *Through a given point there is one and only one perpendicular to a given line.* (6)
4. *Through a given point there is one and only one parallel to a given line.*
5. *A straight line and a circle cannot have more than two points in common.*
6. *Two circles cannot have more than two points in common.*
7. *The sum of two sides of a triangle is greater than the third side.*
8. *The shortest distance from a point to a line is measured along the perpendicular from the point to the line.*
9. *All straight angles have the same measure (180°).*
10. *Vertical angles are equal.*
11. *Two triangles are equal if two sides and the included angle etc.* (1).
12. *Two triangles are equal if a side and the adjacent angles etc.* (2)
13. *Two triangles are equal if three sides etc.* (5)
14. *Two right triangles are equal if the hypotenuse and an adjacent angle, etc.* (9)

* The numbers in parentheses refer to the theorems listed in Document 108, pp. 9-24, of the College Entrance Examination Board.

15. *Two right triangles are equal if the hypotenuse and a side etc.* (10)

I should be pleased if sometime we could broaden assumptions 11, 12, 13 so as to include the corresponding three theorems on similar triangles. The three assumptions would cover six theorems (as given now), equality appearing as a special case in which *k*, the factor of proportionality, equals 1. Since this factor of proportionality can, in general, be any real number, whether rational or irrational, there would be no need to make explicit mention of the incommensurable case at the beginning of Book III.

I should be inclined to take the usual propositions on parallel lines cut by a transversal as assumptions also.

Assumptions for Book II

16. *In the same circle, or in equal circles, equal central angles have equal arcs, and equal arcs have equal central angles.* (37)
17. *In the same circle, or in equal circles, two central angles are proportional to their corresponding arcs.* (38)

It may be desirable to postulate also the theorems concerning equal arcs and chords.

Assumptions for Book III

18, 19, 20. *The three theorems on similar triangles if these have not been included in assumptions 11-13.* (59-61)

21, 22. *If two polygons are similar, they can be divided into triangles which are similar and similarly placed; and conversely.* (70, 71)

Assumptions for Books IV and V

23. *The area of a rectangle is equal to the product of its base by its altitude.* (72)
24. *Regular polygons of the same number of sides are similar.* (82)
25. *If the number of sides of a regular polygon inscribed in a circle be increased indefinitely, the apothem of the polygon will approach the radius of the circle as its limit.* (84)

In this list of assumptions I have included ten starred and six unstarred theorems from the syllabus of the College Entrance Examination Board (Document 108, nos. 1, 2, 5, 6, 9, 10, 37, 38, 59, 60, 61, 70, 71, 72, 82, 84); and I have indicated that the list might be even longer. I see no need for keeping the theorems on inequalities in Books I and II, nos. 26-29 and 53-56 in Document 108. Eu-

clid used them to support his attack on incommensurables; but we, if we fight this foe at all, prefer to use gas or attack from the air. Nos. 13, 16, 17 are little more than exercises, and nos. 85, 87, 88 are bound up in definitions of the circumference and area of circles. All these could be stricken from the syllabus with little loss, leaving about two-thirds as many theorems as at present.

Our enlarged list of assumptions dismisses the method of superposition from the beginning of demonstrative geometry. If at the end of the year the list of assumptions is reconsidered with an eye to reducing its length, it will then be clear why we are attempting to prove the obvious, and the method of superposition will take its proper place.

The foregoing assumptions rest of course on certain definitions and undefined terms. The terms *point, straight line, distince between two points, angle between two lines* ought to be accepted without definition; and also the ordinary logical connectives such as *is, are, not, and, or, but, if, then, all, every.* The undefined term *straight line* includes the idea of infinite extensibility.

The definitions are sufficiently obvious and need not be listed here. The definition of *straight angle* includes the idea "if the sum of two adjacent angles is 180°, their exterior sides form a straight line." The circle can be so defined as to include the notion of the uniqueness of the circle which has a given center and given radius. And we have already seen that the definitions of circumference and area of a circle can include the ideas set forth in propositions 85, 87, 88 of Document 108. If all the propositions commonly proved by means of superposition are taken as assumptions at the outset, there is no need to stipulate that "a figure can be moved about in space without altering its size or shape" or that "superposition shall be a test of equality." In any event it is more in the spirit of the congruence and similarity theorems to phrase the statement as given in the *British Report on The Teaching of Geometry in Schools* (p. 35), as follows: Any figure can be reproduced anywhere, either exactly, or on any enlarged or diminished scale.

There are those who believe that certain parts of trigonometry belong in demonstrative geometry. Granting that trigonometry is a special topic in geometry, I believe that any extended treatment of trigonometry is inappropriate in the midst of a course which is primarily concerned with demonstration. But one can hardly object to

the incidental use of trigonometry, algebra, or the elementary methods of analytic geometry, or any other topic, so long as there is no serious interruption of the main theme—demonstration. The claims of certain significant portions of higher geometry and of the principles of perspective ought also to be considered.

Whatever may be the subject-matter of a revised course, it will probably not be logically complete. This is of slight moment so long as both teacher and pupil are aware of it. The important thing is that the pupil shall gain an appreciation of what constitutes a logically complete system and shall learn to distinguish a good argument from a bad one.

I have already indicated how the list of assumptions at the beginning of geometry might be increased with corresponding diminution of the number of "book-theorems" for the proof of which the student will be held responsible. These proofs will be in general the same as before, but there will be fewer of them. Some theorems, notably those concerning inequalities and incommensurables, have been omitted.

Having relegated so much of the usual threshold of geometry to "assumptions," how then do we begin? Either with many simple "originals" based on our substantial list of assumed propositions; or else with "book-theorems" concerning isosceles triangles, parallels, and parallelograms. I tend to favor beginning with originals in order to show at once what we mean by a proof in geometry. If we begin with isosceles triangles we shall be able with a little care to handle the troublesome converse without resort to superposition or the indirect method. The whole subject of parallels, in both two and three dimensions, makes great use of the indirect method. The theorems on parallelograms are straightforward enough, however. In any event we ought at the outset to exhibit some good specimen proofs of reasoning in geometry which shall serve as patterns for the pupils to follow. These patterns would be equally serviceable, whether they stand as proofs of book theorems or originals.

For basic theorems in the first year of demonstrative geometry I suggest the following. I realize that I am proposing a good many changes. There is no need to adopt these—or alternatives proposed by others—all at once. I am wholly in sympathy with the idea that changes ought not to be made any faster than the traffic can stand. In any case, however, we ought to have before us a definite goal toward which we are tending, even though we approach it by easy stages.

Propositions for Book I*

1, 2. Isosceles triangles (3, 4)

3, 4. Two loci, so worded as not to involve the word "locus" or the idea of a moving point. "All points equidistant from the end-points of a line-segment, and no others, lie on the perpendicular bisector of the line-segment." "All points equidistant from two intersecting lines, and no other points, lie on the pair of lines bisecting the angles formed by the given lines." (30, 31)

5. If a line meets one of two parallel lines, it meets the other also.

6. Two lines parallel to a third line are parallel to each other.

7, 8. Two lines . . . perpendicular . . . are parallel; and a line perpendicular to one of two parallels . . . (7, 8)

9. A transversal meets each line of a system of parallels at the same angle. (11, 12)

10. If a transversal meets two (or more) lines at the same angle, the lines are parallel (14, 15). I have noted above that these propositions on parallels would be good recruits for our list of assumptions.

11-15. The group of theorems on parallelograms (18-22).

16-18. The group of theorems on the sum of the angles of a triangle (23-25).

19-22. The four "concurrence" theorems (32-35).

I should hesitate to make reference to analogous theorems in three dimensions until the idea of a proof is well established. In connection with the congruence theorems (taken as assumptions) it would be pertinent to consider for a moment the possibility of corresponding theorems on a sphere. It would be illuminating also to show that the definition of parallels as "two lines that cannot meet" requires further qualification if it is not to include skew lines in space; while the statement that two lines are parallel if each is parallel to a third line is valid both in two and three dimensions. It would be instructive to elicit from the pupils the statements that "two lines perpendicular to the same plane are parallel," that "both of two parallels are perpendicular to a plane provided one of them is," that "if two parallel planes are cut by a third plane, the lines of intersection are parallel," and perhaps some others closely related to these; and to lead the

* The numbers in parentheses refer to the theorems listed in Document 108, pp. 9-24.

pupils to consider also the conditions under which three planes intersect (two at a time) in three lines, in two lines, or have no point in common. It would be difficult to introduce a class to the concept of locus without making pertinent extensions to three dimensions. But these considerations, valuable in themselves for the insight which they give into spatial relations, do not lend themselves readily to further development in the form of easy exercises; and there is always danger that they may be allowed to intrude too early before the real spirit of the logical aspect of the course has been established. These objections tend to balance each other: the pertinent three-dimensional material at this point is slight and can be so treated as to add meaning to the course without serious interruption. But it is necessary for the teacher to keep the goal clearly in mind.

Propositions for Book II

23. One and only one circle can be drawn through three non-collinear points (36).
24. Equal arcs have equal chords; and conversely (39). These theorems might well be added to our list of assumptions.
25-27. The three theorems (40-42) concerning chords and diameters. These are pretty obvious also, and easy enough to be taken as exercises.
28-31. The four theorems on tangents (43-46).
32-37. The six theorems on the measure of angles (47-52).

It would be appropriate to call attention here to the nature of the intersection of a plane and sphere. But the majority of the references to three-dimensional situations come more naturally in connection with later work in the course.

Propositions for Book III

38, 39. A line parallel to one side of a triangle divides the other two sides proportionally; and conversely (57, 58). These can be proved without reference to incommensurables by means of the assumptions concerning similar triangles.
40. The segments cut off on two transversals by a series of parallels are proportional (64 and 22).
41-47. The usual theorems (62, 63, 65-69) concerning the bisector of an angle of a triangle, the altitude on the hypotenuse of a right triangle, the Pythagorean relation, the products of the segments of chords and secants, and the perimeters of similar polygons.

In connection with the assumptions concerning similar triangles we ought to examine the possibility of analogous theorems on a spherical surface. If two sides of a spherical triangle are doubled, while the included angle remains the same, will the third side be doubled also? And how about the other two angles?

The proof of the theorem that two lines are cut into proportional segments by three parallel planes is essentially plane geometry. The same is true of the theorems concerning planes parallel to the bases of pyramids and cones, which divide the lateral edges, slant height, and altitude proportionally and yield sections similar to the base.

It is easy to apply the Pythagorean theorem to the diagonals of cubes, boxes, houses with gable roofs; to computing the altitude of a square pyramid when the lengths of all the edges are given; and to finding the length of the longest straight wire that will fit inside a tomato can of given diameter and height. It is an easy exercise in numerical trigonometry of the right triangle to find the angle between the diagonals of a cube or box.

Propositions for Book IV

49-53. The six theorems as at present (73-77).

The formulas for the lateral areas of prisms and pyramids are derived by the methods of plane geometry. These, together with the formulas for the volumes of prisms and pyramids (which can be accepted without proof), afford opportunity for many brief and significant excursions into three dimensions. The angle at the vertex of a cone of given slant height and circumference can be easily computed by means of very elementary trigonometry.

Propositions for Book V

54-58. The usual theorems on the circle and regular polygons (78-81, 83).

59, 60. The circumferences of circles are proportional to their radii; their areas are proportional to the squares of the radii (86, 89).

Again, the lateral areas of circular cylinders and cones are found by the methods of plane geometry. Numerical exercises should introduce the student also to the formulas for the volume of cylinder, cone, and sphere, and for the area of the sphere, and to the relation between areas and volumes of similar solids. It is a fact worth knowing that the shortest line that can be drawn on the surface of a sphere, connecting two points of the sphere, is an arc of a great circle.

The usual work in loci and constructions needs almost no modification. It would probably relieve considerable tension in dealing with loci if we agreed to demand two-way proofs only in the case of seven fundamental loci, and to make all other locus proofs dependent on these, so that a simple statement referring to the proper fundamental locus would constitute the whole of the formal proof.

It will be noted that I propose the elimination of thirty or more theorems from the plane geometry syllabus, one-third to be dropped outright and the other two-thirds to be kept as definitions or assumptions and used in support of original exercises. From solid geometry I propose the addition of twenty-eight theorems, for only seven of which the student would be asked to give formal proofs. These seven are in reality nothing but exercises in plane geometry and would presumably be on the order of unstarred theorems. That is, the proof of one of them might conceivably appear on one of the Board's examinations from time to time in the status of an easy original. The other twenty-one belong in the list of assumed propositions and are included for their intrinsic worth as an important part of the instruction and as a foundation for mensurational work in three dimensions; the student would not be asked to prove them on Board examinations. These twenty-eight theorems are those numbered as follows in the section on Solid Geometry in Document 108, pp. 17-24: 12, 10, 11, 17, 30, 32; 64; *22, 47, 56; 41, 48,* 44-46, 51, 52; *54, 57,* 55, 59, 60-63, 90-92. The seven for which proofs may be demanded are printed in italics.

I appreciate the great difficulty of establishing a logical sequence of significant theorems in three-dimensional geometry, however brief, in a first year of demonstrative geometry, and I have not tried to do so. I have hoped that by widening the application of the work in mensuration to include three-dimensional figures we might induce a three-dimensional point of view which would persist throughout the course. The materials for this purpose are taken mainly from Book VII; the methods are the methods of plane geometry applied to plane figures in a three-dimensional environment; the application involves an understanding, intuitively acquired, of the relations of lines and planes in space. An elaboration of this idea, though more extensive than is contemplated here, may be found in the appendix to Document 108, prepared by Professor W. R. Longley of Yale.

We ought not to leave this subject without giving thought to the second course in geometry, now rated as half a unit, which would

follow. A reconsideration of the theorems assumed at the beginning of the first course would give opportunity for discussion of superposition, the indirect method, and incommensurables, some mention of which—the indirect method especially—must have been made in the earlier course. The theorems on inequalities so light-heartedly dismissed from the plane geometry have an important place in connection with solid geometry: we need them to prove that the sum of the angles of a spherical triangle is greater than 180° and less than 540°. On the whole, however, there is little need to revive the thirty book theorems discarded from plane geometry. But more work on two-dimensional originals would be appropriate, especially if some of it could refer to simple yet significant topics from "modern geometry." Our liberal use of three-dimensional mensuration has taken the heart out of Book VII, at least so far as any logical treatment is concerned. Moreover, in recent years our schools have made little serious effort to establish the mensuration of solids on a firm foundation of logic, a tendency which indicates little success for any plan to reinstate it. There remain the strictly logical treatment of lines and planes in space (Book VI), and of the sphere and spherical polygons (Book VIII); and also the continuation of the three-dimensional mensuration to complete the program outlined by Professor Longley in the appendix to Document 108. It would not be difficult to construct a half-unit course from this material.

At present we have "a highly theoretical plane geometry . . . required of all students, while an extremely practical solid geometry is elected by only a few" (H. E. Webb in the MATHEMATICS TEACHER, December, 1919). Professor Tyler of the Massachusetts Institute of Technology suggests a realignment "according to difficulty rather than dimension." I have tried to show how such a realignment might be effected. I believe that casual use of the methods of algebra and numerical trigonometry, and intermittent, but continual, allusion to solid geometry can be introduced into the first-year course in demonstrative geometry without destroying its essence, which is demonstration, while insuring also that no pupil shall quit our modern academy in ignorance of an important aspect of geometry—the three-dimensional domain in which we live and move and have our being.

Merlin and Viviane

By James B. Shaw
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Once upon a time many, many years ago there lived the wisest man in the world. Of stately carriage and dignified mien, he was the center of all eyes wherever he appeared. He was so wise that he knew all things and could do the most marvelous works. He lived in a magnificent castle of his own building, for an he wished, a thousand gnomes and elves appeared, and straightway laid stone upon stone faster than eye could follow, so that in only a few moments of time there arose a stately structure covered with the most fascinating towers and spires, parapets and crenellated lace-work. Intricate designs of colored glass filled the windows, and on the walls were traceries of purple and gold, scarlet and emerald. No one could read their meaning, except the wise man himself, for they were really statements about the laws of the universe. When he traveled he need only wish for a carrier, and wonderful birds with long white wings came to take him upon their backs whither he would go; or sometimes the winds would seize him gently and then with a dizzy rush would set him down at his destination. More surprising still, if he desired to live in the world behind the mirror, a simple wish transported him into a new realm that mortal man had never explored before. If he longed for music he waved his hand, and out of the ether came most entrancing harmonies, weird and keen beyond mortal music, harmonies never expressed on the scale of earthly instruments. No octaves were in these compositions, no major thirds, but minor chords which sounded once and were not repeated in the cadenced melody. Indeed he was able to produce music of such complicated form that each note was itself the whole sequence of an endless composition, and as the symphony flowed on, at every moment the ear was charmed by an infinite symphony given instantaneously. At his command the landscape would bloom with flowers of most intricate design. In ordinary flowers the parts are arranged in a winding cycle, but in his flowers there were cycles of cycles. The patterns they made were a delight to the eye. The units that were repeated in many cycles were them-

selves more and more complicated patterns which could not be untangled into simpler ones. Did Merlin feel the need of diversion, a wave of his wand would bring before him a lovely garden filled with beautiful knights and ladies who would produce dance transformations of a most bewildering variety. In many of these they would be arranged in two opposite sets, and these would interchange over and over but always maintaining their part in the set to which they belonged. In other transformations the dancers would gradually get farther and farther away, becoming smaller and smaller, until having reached a faraway boundary, they would return by a series of the reverse kind.

Merlin was more marvelous still, since he could read the messages from the stars and knew where they would be found in all time to come. The distant nebulae, 72,000,000 light-years away were friends of his as they sped along at more than 7,000 miles a second. And he knew the secrets of the little systems of points called atoms, from which quivering waves spread out through the visible universe by trillions and trillions. He could see the fine silvery cosmic dust flashing through space in all directions and could penetrate to the hearts of the tiny particles whence it darted. Merlin could re-create that which had once happened in the world and make it live again with its actors going through their parts, and he could pass through the dim vistas of the future and bring before him in his Joyous Garden that which is to come. He knew himself as part of that vast organism called the living universe, and he knew himself as also a vast living universe, and could trace the life-histories of the infinitely minute creatures which in him lived and moved and had their being. He could even pass below the realm of that which can be measured, into the realm of the infinitesimal, where an endless series of realms lie within realms. And he knew the magic spell which would keep a process of change going on forever. No material thing held its secret from his sight, and he could create an unlimited wealth of new forms that material things might take.

Merlin was consulted by kings and emperors. When they wished to build bridges, canals, tunnels, or cathedrals, he was the chief engineer. He designed their factories, the machinery which ran inside, then the means of transportation. Into his own realms few could pass, so that for them more ordinary means had to be devised. He told them how to put up their telephone lines, and their wireless broadcasting

antennae. He built airplanes and submarines. He located earthquake foci and predicted the storms that would appear. He managed finances, analysed statistics, and solved problems of population. In fact he was indispensable in all the affairs of the nations.

There lived in the same land as Merlin a fairy named Viviane. She was the daughter of a great scholar and a fairy that lived in one of the lakes. From her father she inherited a very practical nature, and was much given to having things sorted and put away in various places where they could always be found. She used to sit by the lake and dream of the days when she would have an establishment of her own, and her mother used to come up out of the depths of the lake and talk to her and tell her the way to accomplish her purposes. Her mother told her about Merlin and all his wonderful knowledge and skill. And she also told Viviane that when she met Merlin she should undertake to cause him to fall in love with her. Then when the time was ripe she should coax him to reveal to her the secret magic by which a man could be bound fast forever to stay with a woman. Viviane was very ambitious and the program pleased her vanity also. She was attractive and many suitors had tried their luck with her but all had failed to interest her permanently.

In the course of time Merlin rode through the woods of Broceliande, and stopped at a spring to refresh himself with the cool water. After drinking he saw Viviane sitting on the other side of the spring looking at him, and he spoke to her. She asked him who he might be, well knowing however that he was Merlin, though he was disguised as a traveling scholar. He replied that he was a student. She inquired what he studied, and he replied that his was the art of magic. She begged him to show her some of his wonderful powers. He graciously consented and traced his mystical characters upon the grass. The forest glade was at once filled with the most wonderful things Viviane had ever beheld, and she was charmed with their beauty. She begged for the secrets of his magic and he promised that sometime he would reveal them to her. Then he waved his hand and everything vanished but the garden. He went on his way, but a year later passed once again that way to keep his tryst with Viviane.

He found her waiting for him, with an exquisite banquet for two prepared. But the viands, dainty as they were, were somewhat wasted, for he was really becoming much in love with her. She however was, as always, calm and placid as the lake in which her mother

lived. This time she learned some of the secrets of magic, but it was not until the next time Merlin came, when the wild roses were blooming, that she succeeded in finding out the magic charm which would bind him in her power. Under her persuasion he taught her the enchantment that would bind him to her forever. And she used the charm, building for him a tower of air, weaving in the blue of heaven, and wrapping him in fabrics aërial, ethereal, clear, crystalline, and rare.

"Aërial, ethereal,
And crystal clear as glass,
Or mirrored surface of the pool
Before the storm clouds pass,
The elfin charm drave up the air
And scarcely stirred the grass."

"Aërial, ethereal,
Athwart the gleam and glow,
The airy charm wheeled slowly up,
And slowly, and more slow,
'Lo, I am Merlin,' spake a voice,
'And thus our glories go!' "*

It was Merlin who created Arithmetic. He started by creating the system of natural numbers, perhaps to account for the various rhythms that impressed him in his daily life. Then he created the rational numbers. These were not included in the system of natural numbers, were not to be found in nature. They were the product of a magician's wand. Then Merlin caused the irrational numbers to come into being. These were not derived from the rational numbers, for they depend upon a notion which the rational numbers do not need, namely that of *limit*. Beyond these Merlin has created the non-Archimedean numbers, which, like irrationals, are not found in the world of nature. That Merlin will not create some day other numbers, who would believe? That is, provided he be not really spell-bound by Viviane.

Merlin waved his wand and created algebra. This happened when he made new numbers which would satisfy the equations that he studied, most of which had no solutions in the realm of arithmetic. Thence came like a new butterfly the negatives, and after them the four-winged complexes. Many years later in trying to find the sets

* E. Mackinstry

of numbers which would be given as the solutions of equations that contained parameters, he created the entire list of hypernumbers, beginning with quaternions, and continuing to the Dirac numbers of modern physics, and non-associative systems which are yet to be applied to natural problems. Out of arithmetic flows the theory of ensembles, and out of algebra flows the theory of operations. Out of arithmetic comes order and tactic. Out of algebra comes the theory of groups. Arithmetic gives us a universe of distinct objects, while algebra manifolds this world into multitudes of related worlds as if the world of arithmetic had been reflected in magic mirrors over and over. The mirrors themselves have been made more and more numerous till now they are unlimited in number, and the plurality of the worlds has become an infinity.

Merlin looked at the crude drawings of early man and with his power of magic made them exact and refined, then he filled them full of new lines and more and more intricate designs until geometry appeared on the scene. With his discerning eye Merlin saw the lines traced on a cone which he called conic sections, and he studied them a thousand years before they were found in nature. He made other curves, created like the cissoid and the conchoid to enable him to solve his geometric equations. The conchoid furnished the pattern for trisecting an angle, and the cissoid for making cubes which would multiplicate other cubes. He sensed a general procedure of this kind but it was many centuries before he had learned the magic to bring it about. He also used his notion of limit on his curves and areas and volumes producing the geometric irrational. He even had geometric non-Archimedean objects. Then the day came when he created new geometries, whole new worlds of lines and curves; and not content with three-way space, he expanded his universe into space of any number of ways in which he could move and draw his designs. Then Merlin learned a new magic and could make the world of space disappear and become the world of algebra or the world of algebra could become the world of space. In fact it was only one world which appeared in two different forms. The world of arithmetic was also brought into this one world of algebra-geometry at the same time, and the enchantment was very marvelous.

Merlin learned the magic of the limiting processes, and Calculus came to build castles beyond the unending succession of finite steps. Continuity of the arithmetic kind, was set alongside the geometric

continuity; the flow of points on curves alongside the flow of points on other curves; functionality appeared as if magic had created a magician who could turn one object into another object, even of a different kind. The walls and materials of the castles disappeared and the structures made of invariants upon which the geometric worlds were hung became visible. By touching the proper spring a group of transformations rolled the walls around like stage scenery. Even the limiting processes led to new equations for which new functions had to be created and the wand of Merlin was kept active. He had learned now that all the various objects he brought forth were manifestations of the same underlying spirit of mathematics in all their apparently diverse forms.

The way that Merlin created is stated definitely in one place. He says, in brief, that he was trying to show that he could not create a set of functions of the kind called nowadays fuchsian functions. But one night after a rather sleepless time he learned the magic for this kind of creation based upon the hypergeometric series as type. Then he tried to create these functions as quotients of functions similar to the theta functions, and by following the analogy of the elliptic functions, he saw how to do the similar thing for the fuchsian functions. A little later while on a geological trip, he saw the fuchsian functions had turned into geometric properties. After some months while resting at the seashore, he saw that the geometric properties had become the properties of indefinite ternary quadratic forms, and then he was able to create, with this new knowledge, a much more extensive class of the fuchsian functions, with the corresponding theta fuchsian functions. Thus the whole new flora of automorphic functions bloomed in the Joyous Garden. Only one kind was lacking, and one day he found this blooming on a boulevard in Paris. This is one instance in thousands of the way that Merlin comes to develop his magic.

Viviane many centuries ago had interested herself in the prophets, but they were quite indifferent to her charms. Her ideas of hard and fast rules did not appeal to them. Then she was interested in the philosophers. They made much of her, but as she could not control the starting points of their journeys she often arrived at a terminus she did not fancy. She also turned her attentions to the theologians, and here she succeeded in achieving a certain amount of real domination. However, the realm slipped away from her control in the course of the ages. Then she turned to the scientists and

seemed to be destined to be Queen of the world, until the day dawned when it became apparent that Science itself derived most of its resources from the magic of Merlin. So Viviane turned to Merlin with the intention of captivating him.

Now Merlin had not been insensible to the charms of Viviane during his long career as a magician. He had wooed her mildly in the days of Euclid, and through all the centuries following he had offered her no small share of his marvelous worlds. Their ordering and arrangement he had designed to please her and even when he had flown on wings she wist not of, he afterwards arranged paths and roads for her delectation. Not so very long ago he created a realm of her own kind far more extensive than she had ever been able to command by her own resources, and gave it to her as an outright gift. It was situated very close to much of his own territory and indeed part of it was his which he generously added to her domain. It was during this visit to the Joyous Garden that she implored him to tell her the secret which would bind him to her forever. This he did in a moment of enthusiasm, and Viviane has been thinking for some years now what a magnificent realm she rules over, and that Merlin the greatest magician the world has ever had lies bound to her irrevocably.

However, though born of a fairy, Viviane does not have much knowledge of Magic, and she has found that though she can wave the magic wand, nothing ever happens. She can indeed explore the hidden parts of her kingdom, she can go into its hidden caves, she can climb its mountains, but it lies there static, frozen, crystallized in solid glass. The old enchantment of the unexpected, the new, the spontaneous, the unsung music that has not existed before, the strange and exotic flowers of worlds that she knows nothing of—all these are beyond her power. She has even found in reading some of Merlin's books that her own powers are largely due to his fancy, and without that to support them they are elusive, and she fears the day may come when her whole kingdom will prove to be merely a dream:—a dream of Merlin's, which, should he fall into deep enough sleep, would vanish into nothingness. Viviane finds that as she explores the kingdom she frequently gets as the result of all her exertions small and trivial rewards, for without Merlin she has no way of knowing where to find the mines of real jewels, she does not know how to select the direct routes to her goals. She never dreams, no vision ever comes

to entrance her view of the future, she never wanders into the wilderness, for she fears she will get lost. Viviane has security and certainty only upon well laid-out roads which must be followed to their inevitable terminations. If the roads fork she can not determine at all which fork to follow, so is obliged often to retrace her steps and take each one in turn. Indeed she is helpless in the selection of the very means by which she will reach the destination. And now that Merlin is imprisoned in a tower of air she wearies of the prisoner she has made.

But the old legend was incomplete, for there is yet one more part to the story. The secret that Merlin whispered to Viviane is successful only so long as Merlin does not desire to create for himself another and new world. The power to create at will which he possesses, no one can bind, not even Merlin, and Viviane is under an illusion. She may dominate any particular world he chooses to be in if he yields to the spell; but when he vanishes into a new world, he eludes her and all the spells she can ever learn. Being a creative agent and free, Merlin might even fool himself for a time into thinking that he had found the final spell for all enchantment, but he will inevitably, as time goes on, spontaneously and unexpectedly produce a new universe. Whatever order exists in any world Merlin has created, may be inspected and its form made clear, and when Viviane has learned the spell of magic which is the essence of this type of order she has power over it.

Indeed there is emerging today under the magic of Merlin, even though he was thought to be bound, a new world of indeterminacy, a world in which there is order in part, but an ever-changing order, the changes being not under order. This is the magic which creates a fluent world, the ever-new flowers always exhibiting varying form, phantom butterflies resting on their petals, frost-crystals whose designs fade into new designs even while we try to catch them, harmonies whose subtle appeal is witchery that is so elusive it does not register at all. The air in Merlin's tower is blowing again, and the wind bloweth where it listeth. The spell is not dissolving, it is the world itself in which the spell is effective which is vanishing. There is no logic to say what it will be a millenium hence and equally there is no logic to say what it was a million years ago.

Merlin's magic is like the improvisation of a great musician, for while the theme runs on, is developed, expressed in every variation, we have no sooner learned it than there creeps in a new note and a new

chord, and then a new theme, which is added to the old. The whole rises like Abt Vogler's in an etherial cathedral, tower on tower, structure after structure appearing in the music, the work of genii, and becoming grander and more sublime. When one part has been made perfect, a new part already is coming into the symphony, and so on without end. The musician himself does not know whence these arise, even though he follows a system in their development, so he is as much surprised in his divine play as is the hearer. Thus Viviane however successful is obliged to learn the secret spell anew, and in reality she would not have it otherwise. For two phases of mathematics must not be forgotten. One was emphasized not many years ago by Poincaré, in pointing out that if the mathematician stayed inside his ivory tower and merely worked out consequences, he would find himself ultimately engaged in trivialities, going around in circles rather than definitely going somewhere. This phase of mathematics is the one which views the relations of mathematics with the rest of the universe of thought. For instance, consider what the Einstein theory has done for differential geometry. The other phase is that of the new growth of mathematics. It does not consist of artificial flowers of spun glass and wax, kept under a bell-jar. The new postulational systems all originate in some new flowering branch which is alive. One might as well expect to write poetry by assuming a set of words and phrases, and putting them together under a system of rules. Merlin is essential to the Joyous Garden, for without his lively imagination it never would have existed. Mathematics is the daily witness of the creative ability of the human spirit. That it draws necessary conclusions is not a definition, but merely a statement that it is not a whimsical chaos. It has a structural system of invariants, but it is no more a crystallized logical mechanism than is the skylark. And Viviane would not have it otherwise.

So Merlin lies in his tower of air merely waiting for the floating spirit that will beckon him to a new world, a wider universe, the creation of new forms of artistry, a new outblooming of flowers of everlasting beauty.

> Aërial, ethereal,
> And free as winging bird,
> Or tinted orchids high in air
> With petals zephyr-stirred,—
> "Lo, I am Merlin," speaks a voice,
> "And thus is my music heard!"

Some Relations between the High School and the College Curriculum in Mathematics*

By Webster G. Simon
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I was asked to speak this afternoon on "Some Relations between the High School and the College Curriculum in Mathematics." I want to stress the word *Relations*. You and I are employed in the biggest and at the same time the most important industry in America. Any depression in our industry will inevitably cause serious embarrassment to every other industry which requires the slightest intelligence on the part of its members. In order to avoid a depression in the industry of education it is essential that all parts of it coöperate to the fullest extent that is possible. If you in the elementary and secondary schools do not consider sympathetically our problem in the colleges, and if we in the colleges refuse to try to understand your problems, then there will be some very serious weaknesses in the industry of education which will soon cause a disastrous depression.

First of all our problem centers around the question of good teaching. We hear people say that a given individual is or is not a good teacher. When a high school principal or a college dean says that Mr. Smith is a fine teacher, he may base his statement on the fact that Mr. Smith sends few cases of discipline to the office and that he has few failures in his classes. A parent's idea of a good teacher is often that of one who doesn't make John or Mary work too hard. The most common conception of a good teacher is that of a person who enables the student to get a certain minimum of facts at the cost of very little effort. My definition of a good teacher is: "A good teacher is a person who stimulates each student to use his mental equipment to the very limit." Let me use as examples two of my professors in Graduate School. One of them was a good but not an excellent expositor; the other was an excellent one. In the classes of

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the first man I had to be alert every minute, whereas with the second man everything flowed along so smoothly that very little mental effort was needed to follow the lines of proof. From the first man I gained an appreciation of the fact that the present mathematical knowledge of the human race has come as a result of centuries of mental struggles, and by him I was stimulated to make efforts to enlarge my horizon. In the class of the second man I felt that I had seen a beautiful achievement of the human mind, but I was not stimulated to unusual efforts. Today I know four times as much mathematics as a result of the efforts of the first teacher as I do from those of the second. I have asked many students of these two teachers if their experience was the same as mine. Invariably the answer has been in the affirmative. I think we should all agree on which was the greater teacher.

Now the second point in our problem is the enriching the course of our better students. From now on I am going to confine my remarks to the upper third of the students in high school because only a few of the boys and girls of a lower standing go to college. I hope that I shall convince you that there is a real problem of teaching in doing justice to our best students.

The most significant development in college work in the last decade has been the introduction of the tutorial system and of honors' work for undergraduates. Both of these have as their aim the stimulation of the individual student to make the most of his mental equipment. At Harvard every student has a tutor. The job of the tutor is to guide the student in independent reading, and to develop in him the habit of independent thought. Because of financial limitations we cannot go as far at Western Reserve University as they have at Harvard. But for men whose major subject is mathematics, the individual is the unit in terms of which we plan our work.

The first half of our freshman year is spent to a large extent in sifting out the boys who for one reason or another do not make the progress which we think is reasonable to expect. But at the same time we are trying to develop the remainder of the class to a point at which the individual members will be able to carry on a certain amount of independent reading, and thus develop an ability to do real mathematical thinking. Beginning with our sophomores we try to show the students where to get information rather than to give it to them in homeopathic doses. Remember that at this

point I am speaking of the better class of students. President Lowell of Harvard once remarked that in the past the colleges and secondary schools have spent ten times as much on the poor and mediocre students as on the good ones. Now the colleges are trying to give the good student the attention he deserves. Incidentally it is gratifying to find very frequently under this system that somehow a mediocre student catches an enthusiasm for mathematics and becomes a satisfactory one.

Now what relation does all of this have to the high school problem? For those sections which have a high "Probable Learning Rate" individual work can be done subject to financial limitations. But even with a minimum of enrichment the work you can do has a direct bearing on ours. For these good students it is just as important that they understand why they perform a certain operation as how to perform it. A short time ago one of the poorest of my freshmen made the accusation against me, that I am a poor teacher. He said that most of the time he could get an answer, but that I required him to prove that his work was correct. But with this individual if a slight variation from the typical problem worked out in the book is made, he is lost. My contention is that people soon lose the "how" in mathematics if they haven't the "why."

Today we hear a great deal about testing in the schools. But all of you can find out without these refined tests who the students are who constitute the upper third of your classes. You probably know them within a month without giving a single test. Now I should like to see studies made of what can be done to enrich the mathematics of those students. Notable steps have been made in that direction in many places such as Wadleigh High School, New York, in which a small select group is studying the elements of calculus. But it isn't necessary to go that far. With beginning algebra you can make a start. The question of multiplication and division by zero can be made a means of giving the students a better understanding of the nature of multiplication and division. If there is any subject that the students learn mechanically, it is the solution of equations. For the good student a great deal can be done here. Try to get him to see exactly what each step in the work means. Then give him an equation that has no root. The theory of equations offers a wealth of material which can be used to stimulate the thinking of your good students. Take the question of the

operations which lead to equivalent equations, and that of what you can do to a fraction without changing its value. The full understanding of the principles involved is necessary when you come to trigonometry, analytic geometry, and the calculus. But it is surprising how many fairly good students are confused. They use the phrase "Clear of Fractions" when no equation is present, and the word "cancel" without being certain what they are doing.

In the field of geometry there is no need of elaborating on the material available. What is done in geometry has a very direct bearing on college work. I know of no better subject to develop in an individual the habit of accurate use of English. A teacher of mine used to say that the two most important elements of English style are *Clearness* and *Adequateness*. How frequently do we have students say, "I know what it is, but I cannot express it." Certainly a mastery of plane geometry should do a great deal to correct this fault. In analytic geometry and calculus the facts of elementary geometry must be used; but the habit of clear precise thinking is more important. When a boy is laboring with the proof of the formula for the area of a circle in terms of its radius, he is using the methods of integral calculus. He can be told that he is using a method which enables us to find the areas bounded by more complicated curves than the circle. He can be shown that the tangent to a circle can be defined in the same way as the tangent to any other curve, for example an hyperbola. This is connected with the derivative in calculus which enables us to solve many problems in maxima and minima. This work in geometry is an excellent preparation for the theory of limits which is at the basis of higher mathematics. In analytic geometry the student is frequently at a loss as to how to begin a proof, and as to what constitutes a proof. My experience has been that the student who has learned how to use his hypotheses in plane geometry finds little difficulty in getting a start on a problem in analytic geometry; and his conception of a proof helps him here. The ability acquired in plane geometry to think clearly and to express that thought adequately and precisely is of great value in the mathematical work which a boy does in college.

Let me say that conversely a knowledge of analytic geometry and calculus enriches one's ideas of algebra and geometry. It seems clear to me that a teacher of algebra and geometry who has mastered

the fundamental principles of analytic geometry and calculus will be a more satisfactory teacher than one who has not, other things being equal. The process of differentiation and its simpler applications throw a great deal of light on algebra and plane geometry. After studying the locus problem in analytic geometry, a larger conception of that problem comes to one. In fact the farther on we go in mathematics, the more we realize that what we have already learned is a part of a larger truth. From a cultural point of view you in the high schools and we in the colleges can do a great deal at each stage of the good student's development to keep alive in him the fact that there is a unity in mathematics which becomes increasingly apparent as he progresses.

At this point I am going to make an abrupt digression from my subject and make a few remarks about the relation of the high school curriculum in mathematics to other subjects which a student may study in college. During the last twenty years profound changes have taken place in chemistry, biology, economics, and sociology. Formerly the last three subjects mentioned were not mathematical. But with the increasing need for quantitative statements in place of qualitative mathematics is becoming an increasingly important tool in mastering these subjects. The student who expects to get beyond the elements in any one of these fields must have a mastery of algebra which cannot be acquired in one year in high school. No longer can it be said that a physician does not need mathematics. Modern physiology is the physics and chemistry of the human body. Physics for a long time has been a mathematical science; and chemistry is using mathematics more and more. To me it is pitiful to have an individual's progress in medicine retarded because he doesn't know enough mathematics. The high school instruction that produces valuable results for work in the above fields is that which brings out a real understanding of algebra and geometry. Mere mastery of technique is not sufficient. I say this not because it will mean more students in our college courses in mathematics; in fact it should mean fewer students for us. At the present time we have to give instruction in algebra to boys who need it in order to carry on their work in physics, chemistry, and biology. This instruction you can give better than we do. These boys didn't study algebra in the Senior High School in many cases because they were not advised to do it. Therefore I wish that somehow we could get the people in the field of voca-

tional guidance to realize the importance of mathematics for those who ever expect to go beyond the elements of the fields of economics, sociology, chemistry, physics, or biology. Medicine is included as a branch of biology. To meet this need you must build a curriculum which will require that the students in the upper third of the class get a real understanding of mathematics, not a mere technique.

Don't misunderstand me. I am not criticising you any more than we in the colleges should be criticised. Only last spring I wrote a review of a textbook in calculus in which I made the statement that too many of our texts are such that many students who pass have a superficial knowledge of the technique of the subject and little more. The most frequent criticism that I receive from economists, chemists, and biologists is that the students do not seem to be able to use good judgment when they apply their mathematics to a given problem. A friend of mine who is a business economist told me a short time ago that there was very little in the instruction he received in mathematics which caused him to exercise good judgment. These criticisms will be met, I am convinced, only in so far as we are able to stimulate our good students to get an understanding of the "why" in mathematics.

Our college work is being planned more and more to stimulate independent reading and thought on the part of the students. Preparation for this can be started for the good students in the high schools. The success of college work in mathematics and to a large extent in economics, sociology, physics, chemistry, and biology, requires that these students have acquired an understanding of the fundamental principles of algebra and geometry, not merely a superficial knowledge of the technique. If we all in our instruction in mathematics develop good judgment on the part of our students which will lead them when using mathematics to think carefully first and then to act upon that thought, we shall have rendered a great service to society.

Equivalence of Equations in One Unknown

By Robert E. Bruce
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"An equation," says Sir Oliver Lodge, "is the most serious and important thing in mathematics." Whether we approve of the superlatives or not, we are probably ready to grant that equations are "serious and important" and that our teachings about them should be accurate in principle as well as effective in practice.

One of our histories of mathematics suggests that the chief contribution of the Arabs to the solution of equations was the recognition of the application of axioms to the problem.[1] Every advance in human knowledge brings with it new dangers and granting the truth of the statement about the Arabs we are faced with the fact that this so-called contribution has led in our modern textbooks to such "axioms" for the solution of equations as: "If the same number is added to each member of an equation the result is an equation."[2] Whether the result is an equation or not depends of course on whether the one who does the adding makes the equality sign between the two members. And if he does make the sign the result is to the same degree an equation whether the two quantities added are the same or different. What the writer of this so-called axiom had in mind was perhaps something like the following: "If the same quantity is added to both sides of an equation of condition the result is an equation having the *same roots* as the given equation," or in the usual terminology, "an equation equivalent to the given equation." And this statement unfortunately is not universally true, though it is true for numbers and for such other quantities as are usually added in solving an equation. As a matter of fact none of the elementary operations can be performed at random upon the two members of an equation of condition without at times leading to trouble. Using only the two familiar axioms of addition and multiplication we may, starting with any equation, destroy and create roots at will.

[1] *History of Mathematics.* D. E. Smith. Vol. II, p. 436, Ginn and Co.

[2] *Second Course in Algebra* (Revised Edition), Hawkes, Luby, & Touton, p. 16, Ginn and Co. The above quotation is the first of a set of so-called axioms that are used in solving linear equations.

For example: $x^2-5x+6=0$ has the roots 2 and 3. Adding the same chosen quantity to each side we have,

$$x^2 - 5x + 6 + \frac{1}{(x-2)(x-3)} = \frac{1}{(x-2)(x-3)}$$

which is an equation (?) without roots. To derive from this result an equation having the root $-\frac{1}{2}$, for example, it is only necessary to multiply by $2x + 1$. Disguising the result a little we have the following equation derived from the given equation by the use of the addition and multiplication axioms only, yet having no root of the given equation but another root, $-\frac{1}{2}$, selected at random;

$$x^2(2x - 9) + \frac{7x^3 - 29x^2 + 14x + 37}{x^2 - 5x + 6} = \frac{7}{x-3} - \frac{5}{x-2}.$$

If now we attempt to solve this final equation by the obvious method of "clearing of fractions," we get as results not only the root $-\frac{1}{2}$ but also the extraneous solutions 2 and 3 counted twice!

In the matter of adding the same quantity to both sides of an equation even the very elect seem prone to fall. Among the best discussions in English of this general question of equivalence are the following:

Algebra, G. Chrystal: Vol. I, Chapt. XIV.

Monographs on Topics of Modern Mathematics. J. W. A. Young, Ed. Chapt. V, "The Algebraic Equation." G. A. Miller.

In the first of these references the author states that "there are few parts of algebra more important than the logic of the derivation of equations, and few, unhappily, that are treated in more slovenly fashion in elementary teaching." Within a page of this quotation the author has established to his own satisfaction that the addition of "any function of the variables" to both sides of an equation of condition leads to an equivalent equation. The second reference states (Art. 29) that "if the two members of an equation are increased or diminished by the same expression, the resulting equation is evidently equivalent to the original."

We are all accustomed to the fact that "doing the same thing to both sides" doesn't always seem to work well when we raise to a power to remove a radical. If a student has been brought up on the method of "keeping the balance" by the use of axioms, the shattering

of faith when he reaches irrational equations may be as unsettling mathematically speaking as an upheaval of faith in any other line. In particular such a student may well ask: "Are there other operations *that I am in danger of using* that may cause trouble?" The example considered above makes it clear that in all honesty we must reply: "Yes. The simple process of clearing of fractions may get you into trouble, particularly if the equation has been written by some fiend of a textbook writer who is seeking that end."

The sanction of time and usage for the axiom method is probably to some extent due to the constant use of the single word "equation" to denote two things as radically different from each other as the identity and the equation of condition. These two characters of algebra have been associated together simply because they are dressed alike. Essentially they are as different as black is from mushy. The identity is a convenient method of showing optional ways for writing algebraic expressions. The equation of condition is the translation into algebraic symbols of a question the answer to which is the value of x where x is the villain of the story. The student is the detective whose job it is to make the equation divulge the villain. And this task is like "hunting for a needle in a hay stack." For while practically any one of the infinite crowd of the number family can play the part of any character in an identity, the thumb print of x as recorded in the equation of condition limits him, it may be, to a single individual of that vast crowd. Roughly speaking we may say that $\dfrac{x^3 - x}{(x-1)(x+1)} = x$ (1) is always true; and $x^2 - 1 = 0$ (2) is never true. For (1) breaks down for just two values of x, $+1$ and -1; while (2) is true for these two values and for no other values whatsoever. If (1) then is to be called an equation might not (2) well be called an inequality? We do not of course seriously propose stealing the name of another respectable member of the algebraic family in order to rechristen the equation of condition. What we do suggest is that the traditional name should denote more exactly to the student the real character of its bearer. And to accomplish this the contrast between the identity and the equation of condition should be emphasized constantly. Almost any straw in the stack is good enough for the identity, but to find the needle that will satisfy the equation of condition is a game to test the skill of the best.

In no way are the essential differences between these two members of the algebraic family more clearly shown than in their reaction to a diet of axioms. The identity readily assimilates them and at the worst, thereafter, simply refuses to be "satisfied" with a few of the straws of the stack that had previously been acceptable. For example, if $\frac{1}{x-3}$ has been added to both members, the identity thereafter refuses to be satisfied by $x=3$. But we have already seen that even a light lunch of axioms may so change the equation of condition that it thereafter refuses to be satisfied by any of its former roots and demands an entirely new set. The moral of this is obvious: *Don't apply axioms to equations of condition but to identities only.*

It will hardly do to leave this matter here or the teacher will be in doubt as to how the linear and quadratic equations are to be dealt with. This matter will be discussed presently. But let us first see if anything can be salvaged from the ruin above. First for certain fundamental definitions: 1. A *root* of an equation of condition in one unknown is any quantity which when substituted for the unknown reduces the equation of condition to an identity. 2. Two equations of condition are *equivalent* if all their roots are the same. 3. If all the roots of an equation of condition are part but not all of the roots of a second equation, the second equation is said to be *redundant* with respect to the first. 4. If one equation is redundant with respect to a second, then the second is said to be *defective* with respect to the first. 5. The extra roots of an equation that is redundant with respect to a given equation are said to be *extraneous solutions* (not roots) of the given equation. 6. An operation which when performed on any equation *always* leads to an equation equivalent to it is called a *reversible operation.*

The list of those reversible operations that are ordinarily useful in solving equations is given in the following theorems:

Theorem 1. The addition of any constant or any integral function of the unknown to both sides of an equation of condition gives an equation equivalent to the given equation. Similarly for subtraction.

Theorem 2. The multiplication of both sides of an equation of condition by any constant save zero gives an equation equivalent to the given equation. Similarly for division.

Theorem 3. Given an equation of condition consisting of fractions

whose numerators and denominators are constants or rational integral functions of the unknown (*i.e.*, ordinary polynomials); if (1) the fractions are all in lowest terms, and (2) the denominators are relatively prime; then the result of clearing of fractions gives an equation equivalent to the given equation.

If, as often happens, operations other than these are used in solving equations by the balance method then the only safe course is to test the results by substitution before assuming that they are roots. The apparent similarity between the axioms and some of the statements as to reversible operations should deceive no one. The likeness is less than skin deep. Axioms are assumed without proof. These statements as to reversible operations are theorems to be proved. Save for theorem 3 the proofs are simplicity itself. The simpler proofs may be found in the chapter of Chrystal's *Algebra* referred to above and the proof of Theorem 3, in the monograph by Miller. The easier proofs while not particularly interesting show so clearly the relations of equations of condition and identities and also the proper use of the axioms that it is perhaps unfortunate they are not more widely used. Without repeating the materials to be found in the references given above, we may here show the method used in the proofs by establishing a theorem that has been prominent in the earlier parts of this paper.

Theorem: The addition of a fractional function of the unknown to both sides of an equation of condition is not a reversible operation.

Let $F_1(x)$, $F_2(x)$, $F_3(x)$ be rational integral functions of x.

To prove that $F_1(x) + \dfrac{1}{F_3(x)} = F_2(x) + \dfrac{1}{F_3(x)}$ (1)

s not necessarily equivalent to $F_1(x) = F_2(x)$ (2)

Proof:

(I) Let r be any root of (1)

Hence from the definition of a root

$$F_1(r) + \frac{1}{F_3(r)} \equiv F_2(r) + \frac{1}{F_3(r)}\cdot$$

Subtracting the constant $\dfrac{1}{F_3(r)}$ from both sides of this identity (Subtraction axiom) we have

$$F_1(r) \equiv F_2(r).$$

But this is the condition that r is a root of (2). Hence (2) has every root of (1) and must be either equivalent or else redundant with respect to (1).

(II) Let R be any root of (2).

Hence $F_1(R) \equiv F_2(R)$ (3)

R either is or is not a root of the equation

$$F_3(x) = 0.$$

(a) Suppose R is not a root of $F_3(x) = 0$

Then $F_3(R) \not\equiv 0$ and we may add the constant $\dfrac{1}{F_3(R)}$ to both sides of (3)

Thus we have

$$F_1(R) + \frac{1}{F_3(R)} \equiv F_2(R) + \frac{1}{F_3(R)} \qquad (4)$$

But this is the condition that R is a root of (1) and under hypothesis (a), (1) and (2) are equivalent.

(b) Suppose R is a root of $F_3(x) = 0$

Then $F_3(R) \equiv 0$. Thus condition (4) contains an indicated division by zero and R is not a root of (1).

Thus under hypothesis (b), (1) is defective with respect to (2).

Hence as the process of adding a fractional function of the unknown to both sides of an equation may possibly lead to a defective equation the process is not reversible. Q.E.D.

As to the solution of linear and quadratic equations, it is obvious that the teacher must at any cost avoid a plausible lie as a substitute for an inelegant truth. Fortunately there is but one point in these solutions where we find it necessary to use a non-reversible operation, *viz.* in extracting the root in solving a quadratic. And there the situation is saved by the use of the double sign. Concerning the main question we may say that a defense of the methods used by an appeal to axioms is plausible but untrue, defense by means of the theorems of equivalence is true but so cumbersome as to be hardly plausible to the student, defense by means of an appeal to the definition of a root is both plausible and true. The student must admit that $\dfrac{-b}{a}$ satisfies $ax + b = 0$ and that $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ satisfies

$ax^2 + bx + c = 0$. For most students of elementary algebra is not this the best defense? For the exceptional student who presses the question, "But why do the methods I use lead to the roots?", the answer will of course have to be an honest meeting of the situation—a suggestion perhaps of some of the pitfalls and a little about equivalence.

In any event the student may well receive a much earlier introduction to the methods he will use the moment he leaves linear and quadratic equations. In particular is there any method that is more generally used for solving all sorts of equations than the method of scientific, well directed guessing? Why should this method to which the student will eventually come be taboo in the elementary classroom? The old guessing rule of double false position may be made a good ally in hunting down the villain x. And until he is able to handle more advanced methods this rule combined with simple graphic methods will serve the student very well in determining the real roots of equations of all kinds. The field thus opened is broad and attractive, and moreover the methods suggested depend for their defense only on the definition of a root and not on equivalence.

Willebrord Snell van Roijen

1591-1626

WILLEBRORD SNELL or Snellius, as he is frequently called, succeeded his father as professor of mathematics at Leyden in 1613. Ball says of him "He was one of those infant prodigies who occasionally appear, and at the age of twelve he is said to have been acquainted with the standard mathematical works."* His work was principally in astronomy, physics, and trigonometry. He discovered the law of refraction in optics in 1619, discussed the properties of the polar triangle in spherical trigonometry, and stated the area Δ of a triangle in the formula $1:\sin A = bc:2\Delta$. He is connected with other important mathematicians of the Low Countries by his translations into Latin of Stevin's work on hydrostatics, and of van Ceulen's work called in this edition *Fundamemta Arithmetica et Geometrica* which gave the value of π to 35 decimal places.

* *Short Account of the History of Mathematics*, 1915 ed., p. 254.

Our Public Schools Should Teach Thrift

By Thomas F. Wallace

President National Association of Mutual Savings Banks; President Farmers and Mechanics Savings Bank, Minneapolis, Minnesota

High wages or unemployment—inflation or deflation—alike seem unable to teach very much financial wisdom to the majority of our people in respect to the fundamental principles which underlie the management of their personal affairs. More of our people today may have a superficial acquaintance with shares and stocks, rights and margins, than before; but the education thus gained was at a cost which in most cases was painfully expensive.

Nevertheless the number of men and women of all classes who find themselves on the verge or who are already involved in serious financial trouble, does not seem to lessen. Why is this? Well, if I should hazard a guess, I would say it was largely due to the fact that although our present system of public school education undertakes to teach our boys and girls almost everything under the sun, little or no attempt is made to inform them how to manage money, even as to the simplest plan for the management of their personal and household expenses.

Some communities with public spirited bankers and progressive educational leaders have installed school banks and school savings systems which encourage thrift and illustrate how money creates money where they pay interest on such accounts. When supplemented by intelligent, voluntary instruction from the teachers, these efforts undoubtedly accomplish some good.

Such plans, however, are conducted mainly in the lower grades, and seldom does the teacher receive official credit for work accomplished along this line, and thrift education still more infrequently gains a place in the regular curricula of the schools.

We cannot improve this situation to any great degree by trying to educate the average adult in the management of his finances. Large employers of labor, savings institutions and other organizations for the promotion of thrift have all tried and failed, because experience has shown that we can teach a budget plan to only one out of a

thousand "grown-ups." The habit of planning expenditures to fit incomes must be formed, if at all, during the school years, and this can be done only when adequate courses in budgeting, saving, and spending are included in the regular study courses of our public schools.

These courses should be very simple in the beginning. For the first four grades it probably would be unwise to go beyond a school banking and savings system. Then, at the beginning of the fifth grade, introduce rudimentary instruction in budgeting and spending.

In a town near Boston school officials have followed a plan for two years which seems to work well. Each pupil receives an account book with three headings—"Save," "Give," "Have"—and is taught that any expenditure can be put under one or the other of the last two words. It is interesting to notice that where "outgo" is considered, the educators in this town thought "giving" should have priority in the child's mind over "having."

Never would there seem to be a better time than now to emphasize the importance of thrift and management in personal affairs. In a country like our own the prosperity of the nation depends upon the prosperity of the individual, and this in turn depends upon his or her knowledge of those primary rules, the observance of which means success in personal and home management.

The fact is that today we have no place where the mass of young people may obtain the simplest kind of personal training or education in money or property management. It is also a fact that the lack of this training is enormously costly not only to the individual but to society at large. Practically, there is but one place where such training can be instilled during the habit forming years, the only time it will be effective, and that is as part of the regular course of education in our public schools.

When will our educators awake to the importance of this work? There is no more fruitful field for study and experiment than that which embraces the establishment of satisfactory courses in home and private property management, teaching the millions who yearly graduate from our public schools how to spend their earnings. How much of their income, if they become married, can they afford to pledge toward the purchase of a home? What proportion should go for furnishings? What is the minimum percentage that should go into a reserve fund or be paid out for life, accident or health insurance to take care of emergencies? What is a reasonable rate of return

from an investment where the investor's own brain or brawn is not enlisted in the enterprise?

If our public schools will do this service for our children, they will have rounded out their magnificent work in popular education in the elementary principles governing literacy, morals, and health by introducing similar courses in home and personal affairs management, and will have made a continuously progressive contribution to the happiness, social and political sanity of our people and our nation.

REPORT OF THE ELECTION HELD AT THE ANNUAL MEETING OF THE NATIONAL COUNCIL, FEBRUARY 21, 1931, DETROIT, MICHIGAN

For Second Vice-President, 1931-1933	
Martha Hildebrandt, Maywood, Illinois	124 votes
E. H. Taylor, Charleston, Illinois	99 votes
For members of the Board of Directors, 1931-1934	
Marie Gugle, Columbus, Ohio	146 votes
Joseph B. Orleans, New York, New York	115 votes
Hallie S. Poole, Buffalo, New York	75 votes
Mary S. Sabin, Denver, Colorado	118 votes
C. Louis Thiele, Detroit, Michigan	117 votes
M. Bird Weimar, Wichita, Kansas	94 votes

The following were then declared elected:

Second Vice-President:

Martha Hildebrandt, Maywood, Illinois

Three Members of the Board of Directors for the term 1931-1934:

Marie Gugle, Columbus, Ohio
Mary S. Sabin, Denver, Colorado
C. Louis Thiele, Detroit, Michigan

(Signed) EDWIN W. SCHREIBER
Secretary

DISTRIBUTION OF VOTES ACCORDING TO STATES

Alabama	1	Nebraska	1
California	5	New Hampshire	1
Connecticut	1	New Jersey	5
District of Columbia	1	New York	38
Illinois	48	Ohio	31
Indiana	3	Oklahoma	5
Iowa	9	Pennsylvania	3
Kansas	19	Texas	1
Massachusetts	1	Washington	1
Michigan	65	Wisconsin	3
Minnesota	2		
Mississippi	1		247
Missouri	2		

Stimulating Interest in Mathematics by Creating a Mathematical Atmosphere*

By Mary Ruth Cook
Denton, Texas

Mathematics teachers are forced to admit that the attitude of many high school students toward mathematics has been the feeling that the subject is dull, uninteresting, hard, something forced upon them and to be endured until enough credits have been earned to meet the requirements for graduation. Far too many students have never realized the goal of graduation because of their dislike for the subject and consequent inefficiency in the work. This feeling is largely due to the way in which mathematics has been taught. The zeal for teaching the subject matter of mathematics has been so keen that teachers have failed to keep in mind the necessity for arousing interest on the part of their students in that which is being taught. Mr. Rankin of Duke University very fittingly says:

> Our text books and those of us who teach mathematics are so intent that our students shall acquire a certain amount of technique in manipulating the symbols used in mathematics that we have bounded mathematics on the North by X, on the East by $\sin A$, on the South by $\log X$, and on the West by $\sqrt{-1}$ in about the same way we learned to bound Ohio when we studied Geography. The beauty and power of mathematics to set forth truth is lost sight of.

Mathematics should mean more to the pupil than mere problem solving. If the proper attitudes are being acquired by the pupils, the teacher need have no fear that her pupils will not acquire the necessary "technique in manipulating the symbols used in mathematics."

What better way have teachers of mathematics of portraying this "beauty and power of mathematics to set forth truth" at the same time that they are developing the necessary skills and knowledge than by helping their students to develop in themselves proper emotional attitudes by becoming conscious of the presence of mathematics in things all about them?

* Read before the Mathematics Section of the Texas State Teachers Association November 1930.

Why Should We Create a Mathematical Atmosphere

We should create a mathematical atmosphere in order

1. To help make the child mathematics-conscious, i.e., to help him to recognize the presence of mathematics in all things about him.
2. To serve as an incentive for growth and development in mathematics.
3. To challenge respect and appreciation on the part of the child for beauty, truth, and power of mathematics.
4. To make the child a more intelligent user of mathematics and thus increase his efficiency both to himself and to society.
5. To awaken the casual public to some of the uses of mathematics. (This would be accomplished indirectly through the children and directly through the contact of the public with exhibits, etc., portraying the uses of mathematics.)

How May We Create a Mathematical Atmosphere?

We may create a mathematical atmosphere in the following ways:

I. One of the first means of creating a mathematical atmosphere is by the physical equipment of the mathematics class room.

Does the appearance of your mathematics class room, because of its equipment, bespeak the fact that it is a room in which mathematics is taught, just as the appearance of the science class room tells one that it is a place where science is taught? What is some of the equipment that will lend an atmosphere of mathematics to the class room?

(1) One rather essential piece of equipment for the mathematics class room is a graph board. This can easily be made by having one section of the blackboard marked off in inch squares preferably with green paint.

(2) A second thing that is quite an asset to a mathematics class room is a bulletin board on which the pupils and the teachers may post interesting articles, pictures, graphs, etc., which show the uses of mathematics.

(3) A third means of facilitating the work of students in the mathematics class room is through having such blackboard equipment as yard sticks, blackboard compasses, blackboard protractors, T-Squares, triangles, etc.

(4) A fourth bit of equipment for the mathematics classroom consists of models, a carpenter's level, if possible a transit, an angle mirror, a good slide rule, a decimalized tape, etc.

(5) A fifth thing that lends atmosphere to the class room is the use of pictures of mathematical interest. One inexpensive set of pictures that most any school can afford for the rooms in which mathematics is taught is the set of prints of mural paintings on the walls of the mathematics rooms of the Lincoln School in New York City. These lovely colored prints portray the History of Mathematics. There are three of these pictures together with a short story of them written by Dr. David Eugene Smith which may be purchased from the Bureau of Publications, Columbia University. Prints of the men who have been prominent in the development of mathematics may be had from various sources, and these contribute materially to the mathematical atmosphere in the class room. Recently I visited a mathematics class in the Sherman High School. I was delighted to see on the walls of this room pencil and pen sketches of Thales, Plato, Euclid, The Tower of Knowledge and others of the prints which are coming out each month in the "Mathematics Teacher." The teacher in this room told me that pupils in his classes had made these sketches. What was the value of this? These pupils were helping to make their mathematics room distinctive; they were becoming acquainted, at least with the names of some of the men who have contributed to mathematics, and probably they were gaining some knowledge of what each contributed; they in turn may have been awakened to some lasting interest in mathematics. Another type of picture which is appropriate for a mathematics room is a lovely scene or picture of some beautiful building which shows mathematical forms or examples of symmetry.

(6) Slogans, proverbs, mottoes, and the like bearing upon mathematics is a sixth means of calling attention to the fact that the room is one in which mathematics is taught. In this mathematics class which I visited in Sherman, in addition to the sketches of famous mathematicians there were several such proverbs on the walls. Some of these which I recall are "God Eternally Geometrizes," "Mathematics is the Corner Stone of Successful Business," "The Laws of Nature are the Mathematical Thoughts of God," " 'Ye shall know the Truth and the Truth shall make you free.' Mathematics is Truth." These mottoes had also been made by the pupils in the mathematics classes. If these serve no other purpose than to set the child wondering about such things, they have served a worthwhile purpose.

(7) A reading table in the room with some of the supplementary

reading materials that the children can use in connection with the problems they are working on or for leisure reading is another means of creating a mathematical atmosphere. Some of the books which should be on that table are Smith's *Number Stories of Long Ago;* Week's *Boy's Own Arithmetic;* Abbot's *Flatland;* Dudeney's *Amusements in Mathematics,* also his *The Canterbury Puzzles and Other Curious Problems;* Lick's *Recreations in Mathematics;* Smith's *History of Mathematics,* in two volumes, or Sanford's *A Short History of Mathematics.*

II. A second means of creating a mathematical atmosphere is by making the child mathematics conscious.

H. G. Wells in *Mankind in the Making* says,

> The new mathematics is a sort of supplement to language, affording a means of thought about form and quantity and a means of expression, more exact, compact, and ready than ordinary language. The great body of physical science, a great deal of the facts of financial science and endless social and political problems are only thinkable to those who have had sound training in mathematical analysis. . . . . .

It is just such an appreciation of the necessity for mathematical efficiency together with an emotionalized attitude toward the universal presence of mathematics that we hope to awaken in our students. One of the best ways of accomplishing this goal is by doing interesting problems or projects which will develop the necessary manipulative skill, and which at the same time will be related to things within the interest and daily experience of the children. Some of the ways of doing this are:

(1) First, by the use of booklets. My own Seventh Grade Mathematics class in their study of General Mathematics have decided to keep a sort of class scrap book or diary of the work they do this year. They have finished their first unit which in their text is called "Measurement—Computation." Before starting into the actual work of this unit, they took a series of standard tests on the four fundamentals in integers and fractions. These they called "Inventory Tests" placing the taking of these tests on the same basis as the merchant taking inventory to see how much he had accomplished and where he needed to restock. In order to get a picture of their standing with reference to the possible score, each child made a bar graph of the class scores and the possible score. In connection with this preliminary work in graphs they brought in bar graphs clipped from the cur-

rent issues of newspapers and discussed their significance. In discussing a graph showing the growth of a certain insurance company during the past ten years, one of the children asked, "Do they have a bar predicting what the insurance during this year would be?" Upon being told no, he said, "Well I'll bet if we got a graph that would show this year that bar would be less than the last one because this is such hard times." Was this child getting at the real significance of these graphs? The section in their booklet devoted to these tests they have called " 'Check and Double Check' on the Four Fundamentals." Along with the mounted graphs showing their standing, two members of the group have written an essay setting forth the necessity of checking results and checking up on what one has accomplished giving examples from everyday business and banking.

These children have decided to dedicate their book to "The Makers of Mathematics." The group who are working on the dedication will contribute to the class a little of the history of mathematics by looking up and reporting to the class on some of the "makers of mathematics" and what their contributions have been.

Another group in this class are working up the division of their book which they have chosen to call "Geometric Forms Daily Seen." In this division the pupils have drawn the figures they have studied about, giving the characteristics of each. These drawings are followed by pictures showing examples of these forms as seen in daily use, for example, the triangle in the bridge and in the gable end of the house, and circles in design etc.

"Measuring Instruments in Daily Use" is the title chosen for another division. This group has mounted a page of pictures of the most common measuring instruments such as scales, measuring cups, tea spoons, table spoons, clocks, thermometers, rulers, compasses, etc. This is accompanied by an essay on the constant uses made of measuring instruments.

As one part of "Mathematics in Architecture," two members have written an essay on scale drawings, their use in house building, and have illustrated it by the picture of the plan of a house accompanied by the picture of the house.

Another division that has been suggested by members of the group is "Newspaper Clippings that Require an Understanding of Mathematics for Intelligent Reading."

This is only the beginning of the many things that this group of

children will do in working up their book. I have not attempted to point out the drill in manipulative skills that the children are getting through measuring to get the size for their book, measuring in drawing their figures and in arranging their work on the pages, etc., for the purpose of this paper is to point out the possibilities of stimulating interest in mathematics by creating a mathematical atmosphere. A few quotations from the essays of the children and remarks made by the children while doing their work will answer the question as to whether these children are interested in the work which they are doing and whether they are becoming more mathematics conscious:

"Why, I'd never thought of how much we use measurements."

"People don't stop to think of all the measuring instruments they use daily. Just think, the cook in the kitchen uses measuring instruments morning, noon, and night."

"The most important part of house building is the plan. The architect had to know mathematics to know how to let one inch represent so many feet, and how to get the right shape for the various parts of his house."

"We were doing many interesting things in mathematics; so we decided to make a book which will be a record of our year's work. We had an interesting time and everybody enjoyed it."

"I did not think that people used mathematics so much until we began talking about its many uses in our class."

"If you just think about it, there's mathematics in most everything you see."

(2) A second way of helping to make children mathematics conscious is by the use of posters.

In a similar way to that described in the use made of the booklet, posters may be used to create a mathematical atmosphere that will stimulate interest in mathematics. Some interesting subjects which have been suggested for posters are:

"Mathematics in Business"

"What can Man do with Mathematics?"

"Mathematics, the Key to the Universe"

"Mathematics in the Home"

"Mathematics in Nature"

"Mathematics in Costumes"

"Mathematics in Art"

"Does the Architect Use Mathematics"

The description of an experiment carried out in the uses and making of such posters is given by Olive A. Kee in the *Third Year Book* of the National Council of Mathematics Teachers.

(3) A third way of helping to make the child mathematics conscious is by mathematics exhibits. Posters and booklets together with slogans and the like may be used in the working up of mathematics exhibits. This is also one of the best means of awakening the casual public to some appreciation of mathematics and its many uses.

The stimulation of interest through the development of a proper interpretation of the ever present mathematical atmosphere is dependent upon the teacher who has an everlasting belief in the importance of his subject, a thorough understanding of the subject matter, an ever growing stock of interesting bits of mathematical history, an increasing supply of general information upon which to draw for enrichment materials, together with contagious enthusiasm. If teachers of mathematics will resolve to connect mathematics in every way possible with practical life experiences and will strive to open the eyes of each individual to the great possibilities of mathematics, it is not too much to hope that a different notion of mathematics will be developed; so that its real wealth will be recognized and "instead of seeming to be a wilderness of nightmares and terrors it will be a fairyland of flowers and murmuring brooks." Mathematics will be considered "fascinating and easy to understand, a joy to study, a satisfaction when learned."

Notice to Members of the Council